Sec.2-6 special square matrices（Ref. Riley *et al*, Sec. 8.12）

● If a matrix M satisfies

$\left(M^T\right)_{ij} = M_{ji} = M_{ij}$, i.e., $M^T = M$ ：symmetric matrix

$\left(M^T\right)_{ij} = M_{ji} = -M_{ij}$, i.e., $M^T = -M$ ：anti-symmetric(skew-symmetric)matrix

eg. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is symmetric； $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is anti-symmetric

Note 1：For a symmetric matrix, its element are symmetric with respect to the diagonal. Thus any diagonal matrix is automatically symmetric.

Note 2：The diagonal entries of an anti-symmetric matrix vanish.

$\because$ anti-symmetric $\Rightarrow -M_{ii} = \left(M^T\right)_{ii} = M_{ii} \Rightarrow M_{ii} = 0$

Note 3：For any matrix M, we can always rewrite it as a sum of a symmetric part and an anti-symmetric part：

$$M = \frac{M + M^{\mathrm{T}}}{2} + \frac{M - M^{\mathrm{T}}}{2} = \text{symmetric} + \text{anti-symmetric}$$

cf. For any complex number z, $z = \frac{z + z^*}{2} + \frac{z - z^*}{2} = \text{real} + \text{imaginary}$

Note 4：Even if M and N are symmetric（i.e., $M^T = M$ and $N^T = N$）, their product may not be symmetric, i.e.,

$\left(MN\right)^T = N^T M^T = NM \neq MN$, unless the matrices commute.

Note 5：For any matrix R（probably rectangular）, the products $R^T R$ and RR^T are automatically symmetric.

$\because \left(R^T R\right)^T = R^T \left(R^T\right)^T = R^T R$ and $\left(RR^T\right)^T = \left(R^T\right)^T R^T = RR^T$

Note 6：The inverse of a symmetric (anti-symmetric) matrix is also symmetric (anti-symmetric).

$\because M = \pm M^T \Rightarrow M^{-1} = \pm\left(M^T\right)^{-1} = \pm\left(M^{-1}\right)^T$

● If a real matrix O satisfies

$O^{\mathrm{T}} = O^{-1} \leftrightarrow O^{\mathrm{T}} O = I = OO^{\mathrm{T}}$ ：orthogonal（正交）matrix

eg. $R_{-\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \exp\left(\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$ is orthogonal

Note 1：The product of 2 orthogonal matrices is orthogonal.

Note 2：The inverse of an orthogonal matrix is also orthogonal.

$$\because \ \left(O^{-1}\right)^{\mathrm{T}} = \left(O^{\mathrm{T}}\right)^{-1} = \left(O^{-1}\right)^{-1}$$

Note 3：O must be non-singular！

Note 4：The determinant of an orthogonal matrix must be ±1！

$$\because \ I = O^{\mathrm{T}} O$$

$$\therefore \ |I| = \left|O^{\mathrm{T}} O\right| = \left|O^{\mathrm{T}}\right| |O| = |O|^2 \ \Rightarrow \ 1 = |O|^2 \ \text{ or } \ |O| = \pm 1$$

Note 5：An orthogonal matrix represents a linear operator that leaves the lengths （or norms）of vectors unchanged. That is,

if two column vectors $\vec{y}$ and $\vec{x}$ have the relation： $\vec{y} = O\,\vec{x}$, then

$$(\vec{y})^T \vec{y} = (O\,\vec{x})^T O\,\vec{x} = (\vec{x})^T O^T O\,\vec{x} = (\vec{x})^T \vec{x}$$

Note 6：The columns（or rows）of an $n \times n$ orthogonal matrix constitute the components of n orthonormal vectors！

$$\because \ O = \begin{bmatrix} | & | & & | \\ \hat{e}_1 & \hat{e}_2 & \cdots & \hat{e}_n \\ | & | & & | \end{bmatrix} \ \leftrightarrow \ O^T = \begin{bmatrix} — & \hat{e}_1 & — \\ — & \hat{e}_2 & — \\ & \vdots & \\ — & \hat{e}_n & — \end{bmatrix}$$

$$\text{and } \ I = O^T O = \begin{bmatrix} — & \hat{e}_1 & — \\ — & \hat{e}_2 & — \\ & \vdots & \\ — & \hat{e}_n & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \hat{e}_1 & \hat{e}_2 & \cdots & \hat{e}_n \\ | & | & & | \end{bmatrix}$$

● If a matrix M satisfies

$\left(M^{\dagger}\right)_{ij} = M_{ji}^{*} = M_{ij}$, i.e., $M^{\dagger} = M$ ： hermitian matrix

$\left(M^{\dagger}\right)_{ij} = M_{ji}^{*} = -M_{ij}$, i.e., $M^{\dagger} = -M$ ： anti-hermitian matrix

eg. $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ is hermitian（but anti-symmetric！）

$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ is anti-hermitian（but symmetric！）

Note 1：Any real symmetric (anti-symmetric) matrix is automatically hermitian (anti-hermitian).

Note 2：If M is hermitian, then its diagonal elements must be real;
if M is anti-hermitian, then its diagonal elements must be purely imaginary.

$\because M_{ii} = \left(M^{\dagger}\right)_{ii} = M_{ii}^{*}$ when M is hermitian

$-M_{ii} = \left(M^{\dagger}\right)_{ii} = M_{ii}^{*}$ when M is anti-hermitian

Note 3：If M is hermitian, then iM is anti-hermitian;
if M is anti-hermitian, then iM is hermitian.

Note 4：For any matrix M, we can always rewrite it as a sum of a hermitian part and an anti-hermitian part：

$M = \frac{M + M^{\dagger}}{2} + \frac{M - M^{\dagger}}{2}$ = hermitian + anti-hermitian

Note 5：The product of 2 hermitian matrices is not generally hermitian, unless they commute.

Note 6：For any matrix R（probably rectangular）, the products $R^{\dagger}R$ and $RR^{\dagger}$ are automatically hermitian.

$\because \left(R^{\dagger}R\right)^{\dagger} = R^{\dagger}\left(R^{\dagger}\right)^{\dagger} = R^{\dagger}R$ and $\left(RR^{\dagger}\right)^{\dagger} = \left(R^{\dagger}\right)^{\dagger}R^{\dagger} = RR^{\dagger}$

Note 7：The inverse of a hermitian (anti-hermitian) matrix is also hermitian (anti-hermitian).

$\because M = \pm M^{\dagger} \Rightarrow M^{-1} = \pm\left(M^{\dagger}\right)^{-1} = \pm\left(M^{-1}\right)^{\dagger}$

● If a matrix U satisfies

$U^{\dagger} = U^{-1} \quad \leftrightarrow \quad U^{\dagger}U = I = UU^{\dagger}$ ：unitary（么正）matrix

eg. $\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix} = \exp\left(i\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$ is unitary

Note 1：The orthogonal matrices can be considered as a special（real）case of the unitary matrices.

Note 2：The inverse of a unitary matrix is also unitary.

$$\because \ \left(U^{-1}\right)^{\dagger} = \left(U^{\dagger}\right)^{-1} = \left(U^{-1}\right)^{-1}$$

Note 3：U must be non-singular！

Note 4：The determinant of a unitary matrix must be $e^{i\theta} = \cos\theta + i\sin\theta$！

$$\because \ I = U^{\dagger}U$$

$$\therefore \ |I| = |U^{\dagger}U| = |U^{\dagger}||U| = |U|^{*}|U| \ \Rightarrow \ 1 = |U|^{*}|U| \ \text{ or } \ |U| = e^{i\theta}$$

Note 5：A unitary matrix represents a linear operator that leaves the lengths（or norms）of（complex）vectors unchanged. That is,

if two column vectors $\vec{y}$ and $\vec{x}$ have the relation：$\vec{y} = U\vec{x}$, then

$$(\vec{y})^{\dagger}\vec{y} = (U\vec{x})^{\dagger}U\vec{x} = (\vec{x})^{\dagger}U^{\dagger}U\vec{x} = (\vec{x})^{\dagger}\vec{x}$$

Note 6：The product of 2 unitary matrices is unitary.

$$\because \ U_1U_2(U_1U_2)^{\dagger} = U_1U_2U_2^{\dagger}U_1^{\dagger} = U_1IU_1^{\dagger} = I$$

Note 7：If H is hermitian and U is unitary, then $U^{-1}HU$ is hermitian.

$$\because \ \left(U^{-1}HU\right)^{\dagger} = U^{\dagger}H^{\dagger}\left(U^{-1}\right)^{\dagger} = U^{-1}HU$$

Note 8：If H is hermitian, then $U = \exp(iH)$ is unitary.

$$\because \ U^{\dagger} = \left(\exp(iH)\right)^{\dagger} = \left(\sum_{n=0}^{\infty} \frac{(iH)^n}{n!} \right)^{\dagger}$$

$$= \sum_{n=0}^{\infty} \frac{\left((iH)^n\right)^{\dagger}}{n!} = \sum_{n=0}^{\infty} \frac{(-iH)^n}{n!} = \exp(-iH)$$

$$\therefore \ U^{\dagger}U = \exp(-iH)\exp(iH) = \exp(-iH + iH) = I$$

cf. If θ is real, $e^{i\theta}$ is unimodular, i.e., $e^{i\theta}\left(e^{i\theta}\right)^{*} = e^{i\theta}e^{-i\theta} = 1$.

- normal matrix：a matrix that commutes with its Hermitian conjugate
  That is, $M^\dagger M = MM^\dagger$ or $\left[M, M^\dagger\right] = 0$

  Example：Hermitian matrices and unitary matrices（symmetric matrices and orthogonal matrices in the real case）
  ∵ Hermitian matrix：$MM^\dagger = MM = M^\dagger M$
  unitary matrix：$MM^\dagger = MM^{-1} = M^{-1}M = M^\dagger M$

  Note：The inverse of a normal matrix is also normal.

$$\because \quad \left(M^{-1}\right)^\dagger M^{-1} = \left(M^\dagger\right)^{-1} M^{-1} = \left(MM^\dagger\right)^{-1}$$

$$= \left(M^\dagger M\right)^{-1} = M^{-1}\left(M^\dagger\right)^{-1} = M^{-1}\left(M^{-1}\right)^\dagger$$

- Pauli matrices：$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
  (1) They are hermitian and also traceless.
  (2) Their square equals the unit matrix.
  (3) As a result of these 2 features, they must also be unitary.
  (4) $\left[\sigma_x, \sigma_y\right] = 2i\sigma_z$, $\left[\sigma_y, \sigma_z\right] = 2i\sigma_x$, $\left[\sigma_z, \sigma_x\right] = 2i\sigma_y$
  (5) Any 2 of them anticommute. That is, the anticommutator $\left[M, N\right]_+ \equiv MN + NM$ vanishes.
  (6) As a result, $\sigma_x\sigma_y = i\sigma_z$, $\sigma_y\sigma_z = i\sigma_x$, $\sigma_z\sigma_x = i\sigma_y$